c-Dominating Sets for Families of Graphs

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Abstract

The topic of domination in graphs has a rich history, beginning with chess enthusiasts in the 1850s determining how many queens are necessary to dominate an entire chessboard and continuing to current problems involving computer communication networks, social network theory, and other similar problems. We define a dominating set of a graph G to be a set of vertices of G such that every vertex of G is either in the set or adjacent to a vertex in the set. The domination number for a graph G is the size of a minimum dominating set. Determining the domination number of graphs can prove highly useful in solving many types of problems, and recent studies of dominating sets reflect this.

We focus on describing various families of graphs in terms of bounds on the domination number. Although the computation of dominating sets for arbitrary graphs is an NP-complete problem, it is possible to compute certain bounds on the domination number for certain families of graphs. We examine families of graphs, specifically the family of grids, and determine the bounds on domination number for these families. We compare the domination numbers for the various classes of grids with other common families of graphs.

1 Introduction to Domination

Our work focuses on the concept of domination in graphs, a topic that has been studied extensively in recent decades. We begin with an overview of the relevant definitions along with a series of examples.

First, we review a few graph theory basics that are relevant throughout our work. More on the basics of graph theory can be found in [3], for instance. We define a graph $G = (V, E)$ as a pair of sets, with V the nonempty set of vertices of G, and E the set of edges between distinct vertices of G. The cardinality of V, denoted $|V|$, represents the number of vertices in G, while vertices u and v are *adjacent* if the edge uv is in E. Throughout our work we consider only undirected simple graphs $G = (V, E)$, that is, graphs with no direction on the edges and at most one edge between each pair of vertices. We define the *neighborhood of a vertex* v (also called the closed neighborhood of v), as the set of vertices consisting of v and each vertex adjacent to v: $N[v] = \{v\} \cup \{x \in V : vx \in E\}$. The *degree of a vertex* v, written $d(v)$, is defined as the number of edges incident with v. We denote the *minimum degree of* G with $\delta(G)$. Also, we define $d(u, v)$ as the distance between two vertices u and v in V.

A subset S of V is a *dominating set* if every vertex $v \in V$ is *dominated* by some element of S, that is, every $v \in V$ is either an element of S or is adjacent to an element of S. The *domination number,* $\gamma(G)$ is the minimum cardinality of a dominating set S of V. Finally, the graph G is *c-dominated* if for $0 < c \leq 1$, $\gamma(G) \leq c|V|$. We use the concept of c-domination throughout our work to describe the domination numbers for the graphs from various families of graphs.

1.1 Applications of Domination in Graphs

The study of domination in graphs has historical roots as early as the 1850s when European chess enthusiasts studied the problem of "dominating queens," as described in [4]. These enthusiasts worked to determine the minimum number of queens necessary so that every square on a standard 8×8 chessboard is either occupied by a queen or can be directly attacked by a queen, that is, every square is "dominated" by a queen. It was determined that a minimum of five queens are needed, and this problem can be modeled by finding a dominating set of five queens.

The mathematical study of dominating sets began in earnest in the 1960s, and since that time, dominating sets have been used for many different applications. One of the families of graphs we examine in detail is the family of graphs resembling grids, which we define in more detail in Section 2. A practical application demonstrating the importance of domination numbers in these grids utilizes grids to model city blocks. We let the vertices of the grid represent street corners and the edges represent streets between corners. Then, for example, consider the problem of determining how many police officers should be stationed across the city on street corners so that every corner is visible to at least one officer. We assume that each officer can view the corner on which they are stationed and each corner that is no more than one street block away. The domination number of the grid representing the city would provide the minimum number of police officers necessary so that each corner is visible to at least one officer. Finding a minimum dominating set would provide a description of the street corners where officers should be stationed in order to accomplish this. Clearly many other factors would need to be considered if a city were trying to solve such a problem, but dominating sets could serve as an important tool for decision makers.

Dominating sets can be used to model many other problems, including many relating to computer communication networks, social network theory, land surveying, and other similar issues. Determining the domination number for graphs and finding minimum dominating sets could thus prove very useful. However, finding the domination number of a general graph G is an NP-complete problem [2]. Our work focuses on examining certain families of graphs and finding the exact domination numbers for the graphs from these families.

1.2 Domination in Common Families of Graphs

We start by considering a few examples of domination in common families of graphs. Throughout our work, we focus on families of graphs with relatively small bounds on the domination number. For example, consider the family of complete graphs. A complete graph K_n on n vertices is a graph in which each vertex is adjacent to every other vertex, that is, for every vertex $v \in V$, $d(v) = n - 1$. We see by inspection that for all complete graphs, $\gamma(K_n) = 1$, since each vertex $v \in V$ dominates itself and its $n - 1$ neighbors. Thus the set $D = \{v\}$ dominates the entire graph. Hence, we see that for a complete graph K_n, $\gamma(K_n) = \frac{1}{n}|V| = 1$, and so we say K_n is $\frac{1}{n}$-dominated. In Figure 1, we see a complete graph on 6 vertices with a domination number of 1. In this figure, as well as the remaining figures throughout our work, we use grey vertices to represent the vertices in the dominating set D.

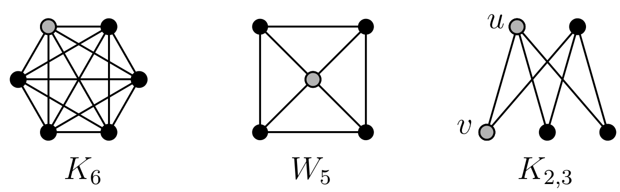

Figure 1: Minimum dominating sets for the graphs K_6, W_5 and $K_{2,3}$

Another family of graphs with a constant domination number of 1 is the family of wheel graphs. A wheel graph W_n is a graph with $n \geq 4$ vertices that contains a cycle of length $n - 1$ and a central vertex v that is not in the cycle but that is adjacent to every vertex in the cycle. Since v is adjacent to every other vertex in W_n, it is clear that v dominates the entire graph. Thus, for all $n \geq 4$, $\gamma(W_n) = \frac{1}{n}|V| = 1$, and so W_n is also $\frac{1}{n}$-dominated. We see the wheel graph W_5 in Figure 1.

Next, consider the family of complete bipartite graphs, that is, graphs in which the vertices can be partitioned into two disjoint subsets U and V, so that each edge connects a vertex from U to V, and every vertex in U is adjacent to every vertex in V. For all complete bipartite graphs $K_{n,m}$ with $n, m \geq 2$ (where U contains n vertices and V contains m vertices), we see by inspection that $\gamma(K_{n,m}) = 2$, since any vertex $u \in U$ dominates each of its m neighbors in V and any vertex $v \in V$ dominates its n neighbors in U. We have then that $K_{n,m}$ is $\frac{2}{n+m}$-dominated, since $\gamma(K_{n,m}) = \frac{2}{n+m}|V| = 2$. For example, consider the complete bipartite graph $K_{2,3}$ in Figure 1.

We see that in these three families of graphs, the domination number is constant in relation to n. In contrast, for a general graph G with minimum degree 1 on n vertices, it has been proven that G is $\frac{1}{2}$-dominated [6], so that the bound on the domination number of G increases linearly with n. For the families of graphs that we examine throughout the remainder of our work, the domination number will also increase with n, though we will prove that the bound on the domination number is significantly less than the general bound for graphs with minimum degree 1.

1.3 Domination in Graphs with Minimum Degree Two

The domination number for graphs with minimum degree two has been explored by William McCuaig and Bruce Shepherd and proven in [5]. The result provides a bound on the domination number of such graphs with only seven exceptions (details on the exceptional graphs can be found in the original paper [5]).

Theorem 1.1. *(McGuaig and Shepard [5]) If $G = (V, E)$ is a connected graph with minimum degree greater than or equal to 2 ($\delta(G) \geq 2$) and G is not a graph of type $\mathcal{B}$ of exceptional graphs, then $\gamma(G) \leq \frac{2}{5}|V|$.*

Thus with this result, we see that if a graph G is connected and contains no vertices of degree 1, and is not one of the seven "bad" graphs in $\mathcal{B}$, the bound on domination number is $\frac{2}{5}|V|$.

2 Domination in Grids

We consider the family of grid graphs, a family of graphs with minimum degree 2 but which has bounds on domination number lower than the general bound given in Theorem 1.1.

2.1 Definition of Grids

For the remainder of our work, we look at the class of grid graphs, and we determine bounds on the domination number based on the size of the grid. Such graphs resemble two-dimensional grids and can be used to model things as important as city blocks, and therefore could be used in applied problems related to city congestion and/or traversal of streets. In formal terms, a two-dimensional *grid graph* is an $m \times n$ graph $G(m \times n)$ that is the *graph Cartesian product* of two paths of length m and n, respectively. Rather than explaining the technical definition of the graph Cartesian product, we explain through an example. Figure 2 illustrates the 3×5 grid $G(3 \times 5)$ on 15 vertices, which is the graph Cartesian product of the path of length 3 and the path of length 5, respectively.

For the purposes of this paper, we define several classes of vertices in relation to grids. Let a *corner vertex* be defined as one of the four vertices of degree two that occurs in the first or the n-th columns of a grid. We define an *outside vertex* as one of the vertices of

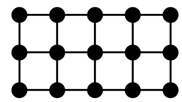

Figure 2: The grid $G(3 \times 5)$

degree three that occurs in the first or the n-th columns of a grid, or in the first or m-th rows of the grid. We define an *inside vertex* as one of the vertices of degree four that occurs in the second through $(n-1)$-st columns or the second through $(m-1)$-st rows of the grid.

2.2 The Case of $G(2 \times n)$

Consider grids of size $2 \times n$ on $2n$ vertices, that is, all grids with two rows and n columns (or, equivalently, n rows and two columns). We will prove that the graph $G(2 \times n)$ has domination number

$$\gamma(G(2 \times n)) = \left\lceil \frac{n+1}{2} \right\rceil.$$

Our technique is to show that the expression provides both an upper and a lower bound. The upper bound argument is constructive, while the lower bound will require a more technical proof.

Lemma 2.1. *The graph $G(2 \times n)$ has domination number satisfying*

$$\gamma(G(2 \times n)) \leq \left\lceil \frac{n+1}{2} \right\rceil.$$

Proof. We give an explicit construction of a set of vertices that dominate the graph $G(2 \times n)$ and meet the bound. We break into cases depending on whether n is even or odd.

Case 1: n is even.

Let D be a subset of V such that D contains one vertex in each odd-numbered column k, alternating between the first and second rows, and one vertex in the n-th column, as seen in Figure 3. That is, if D contains the vertex in the first column in the second row, then D also contains the vertex in the third column in the first row, and the vertex in the fifth column in the second row, and so forth, as well as one vertex in the n-th column in either row. Then, D contains exactly $\frac{n}{2} + 1 = \lceil \frac{n+1}{2} \rceil$ vertices. Let D_1 be the subset of D containing all vertices of D except the vertex in the n-th column, so that $|D_1| = \frac{n}{2}$. For each pair of vertices u and v in D_1, $d(u, v) > 2$, so that $N[u] \cap N[v] = \emptyset$, and each vertex in V is adjacent to no more than one vertex in D_1. The subset D_1 contains one corner vertex in G that dominates itself and its two neighbors, while all other vertices in D_1 are of degree three and dominate themselves and their three neighbors. So, D_1 dominates

$$\left(\frac{n}{2} - 1\right) 4 + (1 \cdot 3) = 2n - 4 + 3 = 2n - 1$$

vertices in G. Then, D_1 dominates the first $n-1$ columns of G, which contain $2(n-1) = 2n-2$ vertices, as well as one of the vertices in the n-th column of G. Now, let w be the vertex in D in the n-th column. We see that w dominates both itself and the other vertex in the n-th column, so that $D = D_1 \cup \{w\}$ dominates G.

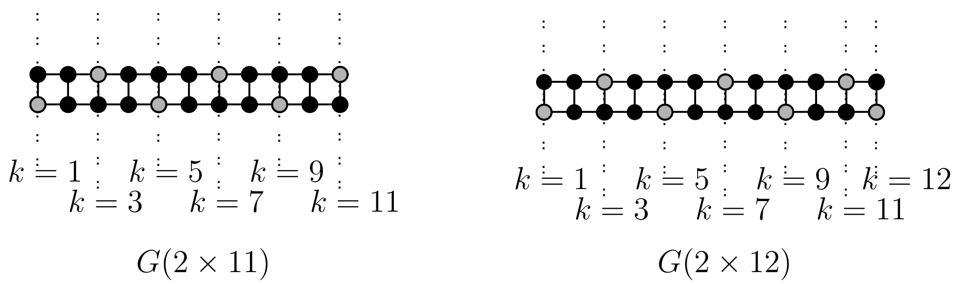

Figure 3: Minimum dominating sets for the grids $G(2 \times 11)$ and $G(2 \times 12)$

Case 2: n is odd.

Let D be a subset of V such that D contains one vertex in each odd-numbered column k, alternating between the first and second rows, as above and as seen in Figure 3. Then, D contains exactly $\frac{n+1}{2}$ vertices. Note that when n is odd, $\frac{n+1}{2} = \left\lceil \frac{n+1}{2} \right\rceil$. Again, for each pair of vertices u and v in D, $d(u,v) > 2$, so that $N[u] \cap N[v] = \emptyset$, and each vertex in V is adjacent to no more than one vertex in D. Since D contains two corner vertices in G and all other vertices in D are of degree three, D dominates

$$\left(\frac{n+1}{2} - 2\right) 4 + (2 \cdot 3) = 2(n+1) - 8 + 6 = 2n + 2 - 8 + 6 = 2n$$

vertices. But G contains exactly $2n$ vertices, and so D dominates G.

By the case analysis above, it follows that $\gamma(G(2 \times n)) \leq \left\lceil \frac{n+1}{2} \right\rceil$.

$\square$

Lemma 2.2. *The graph $G(2 \times n)$ has domination number satisfying*

$$\gamma(G(2 \times n)) \geq \left\lceil \frac{n+1}{2} \right\rceil.$$

Proof. We will prove that no set with less than $\left\lceil \frac{n+1}{2} \right\rceil$ vertices can dominate G. Suppose to the contrary that D is a minimum dominating set with fewer than $\left\lceil \frac{n+1}{2} \right\rceil$ vertices.

Case 1: n is odd.

We are assuming that $|D| < \left\lceil \frac{n+1}{2} \right\rceil$, and since n is odd, it follows that $|D| \leq \left\lceil \frac{n-1}{2} \right\rceil = \frac{n-1}{2}$. All vertices in G have degree of either two or three, so that each vertex in D dominates at most itself and three adjacent vertices. Hence, the number of dominated vertices, $dom(D)$ satisfies

$$dom(D) \leq \frac{n-1}{2} + 3\left(\frac{n-1}{2}\right) = \frac{n-1}{2} + \frac{3n-3}{2} = \frac{4n-4}{2} = 2n-2.$$

But $G(2 \times n)$ has $2n$ vertices, and so D cannot dominate G.

Case 2: n is even.

We are assuming that $|D| < \left\lceil \frac{n+1}{2} \right\rceil$, and since n is even, it follows that $|D| \leq \frac{n}{2}$. All vertices in G have degree of either two or three, so that each vertex in D dominates at most itself and three adjacent vertices. Hence, the number of dominated vertices, $dom(D)$ satisfies

$$dom(D) \leq \frac{n}{2} + 3\left(\frac{n}{2}\right) = \frac{n}{2} + \frac{3n}{2} = \frac{4n}{2} = 2n.$$

This bound is met if and only if each vertex in D is of degree three and the neighborhoods of the vertices in D are all disjoint. However, if D contains only vertices of degree three, then in order to dominate all four corners, both vertices in the 2nd column and both vertices in the $(n-1)$-st column must be in D, contradicting the disjoint neighborhoods of vertices in D. Thus, D cannot dominate G with $|D| < \left\lceil \frac{n+1}{2} \right\rceil$.

By the case analysis above, it follows that $\gamma(G(2 \times n)) \geq \left\lceil \frac{n+1}{2} \right\rceil$.

□

Theorem 2.3. *The graph $G(2 \times n)$ has domination number satisfying*

$$\gamma(G(2 \times n)) = \left\lceil \frac{n+1}{2} \right\rceil.$$

Proof. Follows immediately from Lemmas 2.1 and 2.2. □

Thus, the domination number of grids $G(2 \times n)$ is approximately $\frac{n}{2} = \frac{|V|}{4}$, which is significantly less than the bound proven for general graphs with minimum degree 2. We will see that this is true even with larger grids $G(3 \times n)$ and $G(4 \times n)$.

As we consider grids $G(m \times n)$ with $m > 2$, we quickly see that proving bounds on the domination number of these grids becomes far more complicated than in the case of $G(2 \times n)$. In fact, we will only provide bounds for the cases of $G(3 \times n)$ and $G(4 \times n)$. These bounds and the techniques used in proving them could potentially be used to provide bounds for grids $G(m \times n)$ with $m \geq 5$, but we will not examine these cases in detail.

2.3 The Case of $G(3 \times n)$

We consider grids of the form $3 \times n$ on $3n$ vertices and prove that graphs of this form have domination number satisfying

$$\gamma(G(3 \times n)) = \left\lceil \frac{3n+1}{4} \right\rceil.$$

We use the same technique as with $2 \times n$ grids, first providing a construction of a dominating set for $G(3 \times n)$, and then a more technical proof using strong induction on n to prove that no set with fewer than $\lceil \frac{3n+1}{4} \rceil$ vertices can dominate $G(3 \times n)$.

Lemma 2.4. *The graph $G(3 \times n)$ has domination number satisfying*

$$\gamma(G(3 \times n)) \leq \left\lceil \frac{3n+1}{4} \right\rceil .$$

Proof. We will give an explicit construction of a set of vertices that dominates the graph $G(3\times n)$ and meets the bound. Figure 4 shows such a construction for grids $G(3\times 1)$ through $G(3 \times 9)$.

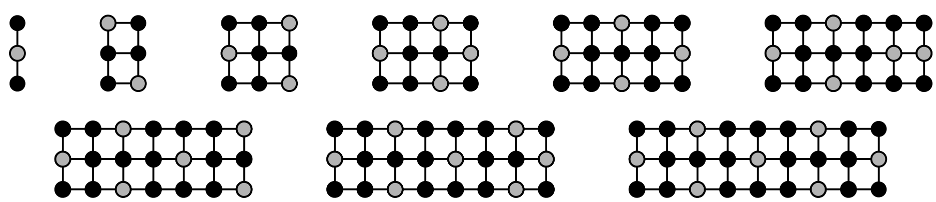

Figure 4: Minimum dominating sets for the grids $G(3 \times 1)$ through $G(3 \times 9)$

For grids with $n \leq 5$ the constructions in Figure 4 suffice. As we consider a general construction for grids with $n > 5$, we refer to the configuration of vertices in D in the construction of the dominating set for $G(3\times 5)$, as seen in Figure 4. The idea for larger grids is to repeat this pattern. By looking at the various congruence classes for n modulo 4, it is not difficult to obtain the lower bound of the lemma.

□

Lemma 2.5. *The graph $G(3 \times n)$ has domination number satisfying*

$$\gamma(G(3 \times n)) \geq \left\lceil \frac{3n+1}{4} \right\rceil .$$

Proof. The proof technique is strong mathematical induction, followed by a careful case analysis. We only provide an overview of the proof, and refer the reader to [7] for the details of the argument. The strong mathematical induction is on n, the length of the grid. That is, we use induction on n to show that $\gamma(G(3 \times n)) \geq \lceil \frac{3n+1}{4} \rceil$. First, we note that $\lceil \frac{3n+1}{4} \rceil = n - \lfloor \frac{n-1}{4} \rfloor$.

Base case: $n = 1$**.**

We see by inspection that $\gamma(G(3 \times 1)) = 1 = \left\lceil \frac{3(1)+1}{4} \right\rceil$, so that the bound holds for the base case.

Inductive step:

Fix an n and suppose that for all values k less than or equal to n, $\gamma(G(3\times n)) \geq \left\lceil \frac{3n+1}{4} \right\rceil = n - \left\lfloor \frac{n-1}{4} \right\rfloor$. We will prove that $\gamma(G(3 \times (n+1))) \geq \left\lceil \frac{3(n+1)+1}{4} \right\rceil = \left\lceil \frac{3n+4}{4} \right\rceil = n+1-\left\lfloor \frac{n}{4} \right\rfloor$.

Let $G = G(3 \times (n+1))$, and let D be a dominating set of G. Then D must dominate a, b, and c, the vertices in the $(n+1)$-st column, in addition to the remaining vertices in G. We now consider the following cases as we consider how a, b, and c can be dominated.

Case 1: $a, b, c \notin D$.

In order for D to dominate a, b, and c, we have that $d, e, f \in D$. Consider $G' = (V', E') = G \backslash \{a, b, c, d, e, f, g, h, i\}$, that is, consider the graph G', obtained by removing the three right-most columns from G, so that $G' = G(3 \times (n-2))$, as seen in Figure 5. Now we consider a set D' that dominates G'. Let D' be $D \cap V'$ plus all vertices u in the $(n-2)$-nd column of G such that u is dominated by some vertex v in the $(n-1)$-st column. If there are m such vertices u, we have that $|D'| \leq |D| - 3 - m + m = |D| - 3$. Then, D' dominates $G' = G(3 \times (n-2))$. By the inductive hypothesis, we know that $|D'| \geq (n-2) - \left\lfloor \frac{n-3}{4} \right\rfloor$. Furthermore, we have that $|D| \geq |D'| + 3 \geq n+1-\left\lfloor \frac{n-3}{4} \right\rfloor \geq n+1-\left\lfloor \frac{n}{4} \right\rfloor$. Thus, the bound holds in this case.

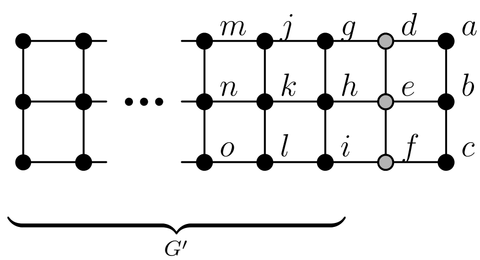

Figure 5: A $G(3 \times (n+1))$ grid with $a, b, c \notin D$

Case 2: At least two of a, b, c are in D.

In this case, all three vertices are dominated by D. Deleting the last two columns of G as in Case 1, we obtain a similar conclusion.

Case 3: Exactly one of a, b, c is in D.

This is the most difficult case in that we must consider two possibilities, either $a \in D$ (or $c \in D$) or $b \in D$, and each will require multiple subcases. However, the argument for each case is very similar to the arguments given in Cases 1 and 2 above, and we omit the details. So, by strong induction on n, we have thus shown that $\gamma(G(3\times n)) = n - \left\lfloor \frac{n-1}{4} \right\rfloor = \left\lceil \frac{3n+1}{4} \right\rceil$. $\square$

Theorem 2.6. *The graph $G(3 \times n)$ has domination number satisfying*

$$\gamma(G(3 \times n)) = \left\lceil \frac{3n+1}{4} \right\rceil.$$

Proof. Follows immediately from Lemmas 2.4 and 2.5. $\square$

We see then that $G(3 \times n)$, a graph on $3n$ vertices, is dominated by $\lceil \frac{3n+1}{4} \rceil$ vertices, so that $G(3 \times n)$ is approximately $\frac{1}{4}$-dominated. As with the families of graphs we have previously examined, this is significantly lower than the $\frac{2}{5}$ bound provided in Theorem 1.1.

2.4 The Case of $G(4 \times n)$

Before considering the general case of $G(4 \times n)$, we consider the grid $G(4 \times 4)$. Two equivalent patterns of vertices in a minimum dominating set for $G(4 \times 4)$ are shown in Figure 6.

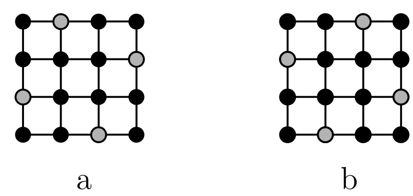

Figure 6: Equivalent minimum dominating sets for the grid $G(4 \times 4)$

Proving that these sets dominate $G(4 \times 4)$ is straight-forward. Counting can then be used to argue that no smaller sets can dominate and that these sets are, in fact, the only minimum dominating sets. We state this precisely in the following theorem.

Theorem 2.7. *The graph $G(4 \times 4)$ has domination number $\gamma(G(4 \times 4)) = 4$. Moreover, dominating sets given in Figure 6 are the only sets of size four that dominate $G(4 \times 4)$.*

Minimum dominating sets can be computed similarly for grids $G(4 \times n)$ with $n \leq 10$. We used the software package Magma [1] to calculate the domination number for these grids, and configurations of a minimum dominating set for each of these grids is shown in Figure 7. We provide the following proof of the general case where $n \geq 10$, using the configuration of vertices in the grid $G(4 \times 4)$ as a basis for the general construction.

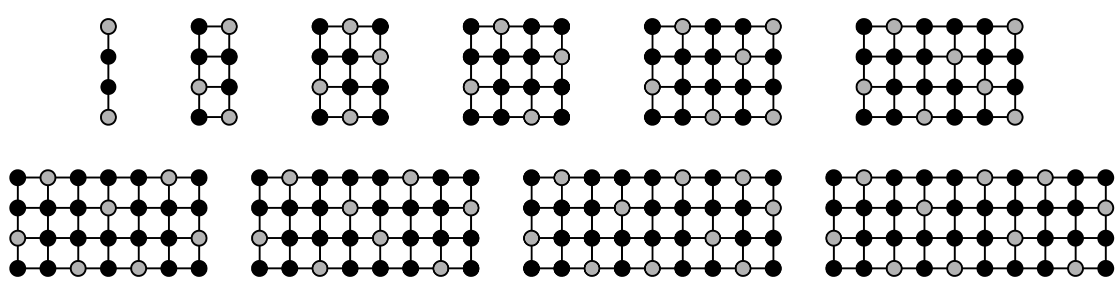

Figure 7: Minimum dominating sets for the grids $G(4 \times 1)$ through $G(4 \times 10)$

For all $n \geq 10$, we can provide an explicit construction of a set of vertices D that dominates the graph $G = G(4 \times n)$ and meets the bound $\gamma(G(4 \times n)) \leq n$. The idea is quite simply to repeat the pattern given by a minimum dominating set for $G(4 \times 4)$ as shown in Theorem 2.7. By considering cases on the congruence class of n modulo 4, one can easily find patterns of vertices that dominate the grids.

To show that $\gamma(G(4 \times n)) \geq n$, we start with a minimal counter-example. That is, we consider the minimum value k so that $\gamma(G(4 \times k)) < k$. After a careful case analysis, this leads to a contradiction, giving us our main result.

Theorem 2.8. *For $n \geq 10$, the graph $G(4 \times n)$ has domination number satisfying*

$$\gamma(G(4 \times n)) = n.$$

Since $G(4 \times n)$ is a graph on $4n$ vertices that is dominated by n vertices, we see that $G(4 \times n)$ is $\frac{1}{4}$-dominated. Thus, as with the case of $G(2 \times n)$ and $G(3 \times n)$, we see that the domination number for grids $G(4 \times n)$ is significantly less than the general bound provided by Theorem 1.1.

3 Conclusion

We have identified several common families of graphs with domination numbers significantly lower than the bound $\gamma(G) \leq \frac{2}{5}|V|$ for general graphs G with minimum degree 2. Though we did not examine larger graphs of size $G(m \times n)$ with $m \geq 5$, the techniques we used to find the bounds for the smaller sizes of grids could be applied to find at least an upper bound for the domination number of larger grids. For example, for a grid of size $G(8 \times n)$, the pattern used to provide a construction for a minimum dominating set of $G(4 \times n)$ could be repeated to create a construction of a minimum dominating set containing $2n$ vertices, thus proving that $\gamma(G(8 \times n)) \leq 2n$. However, we will not examine these larger grids further.

Table 1: Domination numbers of family of graphs

Family	**Notation**	**Domination Number**	**c**
Graphs with $\delta(G) \geq 1$	G	$\gamma(G) \leq \frac{\lvert V \rvert}{2}$	$\frac{1}{2}$
Connected Graphs with $\delta(G) \geq 2$	G	$\gamma(G) \leq \frac{2}{5}\lvert V \rvert$	$\frac{2}{5}$
Complete Graphs ($n \geq 3$)	K_n	$\gamma(K_n) = 1$	$\frac{1}{n}$
Wheel Graphs ($n \geq 4$)	W_n	$\gamma(W_n) = 1$	$\frac{1}{n}$
Complete Bipartite Graphs ($n, m \geq 2$)	$K_{n,m}$	$\gamma(K_{n,m}) = 2$	$\frac{2}{n+m}$
$2 \times n$ Grids	$G(2 \times n)$	$\gamma(G(2 \times n)) = \left\lceil \frac{n+1}{2} \right\rceil$	$\sim \frac{1}{4}$
$3 \times n$ Grids	$G(3 \times n)$	$\gamma(G(3 \times n)) = \left\lceil \frac{3n+1}{4} \right\rceil$	$\sim \frac{1}{4}$
$4 \times n$ Grids ($n \geq 10$)	$G(4 \times n)$	$\gamma(G(4 \times n)) = n$	$\frac{1}{4}$

Table 1 summarizes the different families of graphs we have examined, the domination number associated with each family, and the value of c associated with each family indicating that the family is c-dominated.

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