Faculty Development Grant - Final Report
Summer 2010 Stipend
Determining sets arising from configurations of conics

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My proposal to examine determining sets from configurations of conics was originally written rather broadly since it was difficult to anticipate in the fall of 2009 how this line of research was going to pan out. There are different types of determining sets. In the extremal case, determining sets have a different name: *blocking semiovals.* When I wrote my proposal in the fall of 2009, I would have loved to have titled it *Blocking semiovals arising from...* But this seemed rather ambitious and it appeared, at the time, that the examination of blocking semiovals, or even just semiovals in general, would be very difficult. I am happy to report that I was wrong. While I was not able to classify completely the blocking semiovals arising from conics, I was able to classify completely the semiovals (not necessarily blocking) coming from conics.

The work began fairly early in the spring semester and continued throughout the summer. My colleague, Jeremy Dover, a former NSA employee worked with me. We collected a tremendous amount of data about semiovals, recognized many patterns, and made several conjectures about how conics could be used to construct semiovals. At an international conference in April, we discussed our project with a lot of our colleagues from around the globe and learned of some very helpful resources. In the end, we were able to classify completely the semiovals that can be constructed from unions of conics in the finite projective plane. This is actually a fairly strong result. It is not unusual for me to be able to construct examples, even infinite families, of whatever object I am currently examining. But it is not typical to give a complete characterization as we were able to do. The result was one of the nicest pieces of work I completed in recent years.

I appreciate the funds UMW provided for me to complete this research. The work was written up in the form of a scholarly article during the late summer of 2010 and we submitted the article to the refereed journal *Innovations in Incidence Geometry.* The first page of the submitted article is attached to this report. Moreover, extensions of our results to *blocking* semiovals is already underway.

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Semiovals from unions of conics

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Abstract

A semioval in a projective plane π is a collection of points $\mathcal{O}$ with the property that for every point P of $\mathcal{O}$, there exists exactly one line of π meeting $\mathcal{O}$ precisely in the point P. There are many known constructions of and theoretical results about semiovals, especially those that contain large collinear subsets.

A non-degenerate conic of π, the set of points whose homogeneous coordinates satisfy some non-degenerate quadratic form, is an example of a semioval of size $q+1$ that also forms an arc (i.e., no three points are collinear). As conics are minimal semiovals, it is natural to use them as building blocks for larger semiovals. Our goal in this work it to completely classify sets of conics whose union forms a semioval.

Keywords: semioval, conic

MSC 2000: 51E20

1 Introduction

Let $\mathcal{PG}(2,q)$ denote the finite Desarguesian plane of order q. A semioval in $\mathcal{PG}(2,q)$ is a collection of points $\mathcal{O}$ with the property that for every point P of $\mathcal{O}$, there exists exactly one line meeting $\mathcal{O}$ precisely in the point P. A well-known construction for semiovals is the so-called vertex-less triangle, the set of points formed by the intersection of three non-concurrent lines with the intersection points removed. Many other constructions of semiovals can be formed by taking the vertex-less triangle and then adding and removing certain points in some clever fashion so as to maintain the semioval property. A nice survey of results on semiovals can be found in Kiss [4].

In context of minimal blocking sets, Szönyí [7] looked at collections of conics lying in a common conic pencil, such that their union forms a blocking set, i.e. a set disjoint from no line but also containing no line. Kiss, et. al. [5], later discovered that Szönyí's sets are in fact semiovals. Extending the work in these two papers, our goal here is to characterize all sets of conics in $\mathcal{PG}(2,q)$ whose union is a semioval in the plane. We summarize our principal result here.

Theorem 1.1. *Let $\mathcal{O} = \bigcup_{i=1}^{k} \mathcal{C}_i$ be a semioval in $\mathcal{PG}(2,q)$ that is the union of nondegenerate conics $\mathcal{C}_i$. Then $\mathcal{O}$ is isomorphic to one of the following sets:*

- *a conic (note that this is the only possibility when q is even), or*
- *a union of at most $\sqrt{q}$ conics all lying in a common pencil, or*
- *a union of at most four conics, no three in a common pencil.*

We will prove this result with a thorough case analysis in the following sections. Our analysis will show that each of these cases occurs only for certain values of q. Moreover, we will be able to provide explicit coordinate representations in each case. Our techniques are both algebraic and synthetic.

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